1. Exercises from 5.7/5.8

THEOREM 1. *Let S be an oriented surface in $\mathbb{R}^3$ which is bounded by a piecewise smooth curve ∂S, where the boundary is given an orientation consistent with S, then if $\mathbf{F}(x, y, z)$ is a C^1 vector field defined on a neighbourhood of S in $\mathbb{R}^3$:*

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{x} = \int \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$$

PROBLEM 1. *(Folland 5.7.2) Evaluate:*

$$\int_C y \, dx + y^2 \, dy + (x + 2z) \, dz$$

where C is the curve given by the intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $y + z = a$, oriented counterclockwise as viewed from above

- We will apply Stokes' theorem
- First compute the gradient of $\mathbf{F}$

$$\nabla \times \mathbf{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & y^2 & x + 2z \end{vmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

- Notice that $\nabla \times \mathbf{F}$ is colinear with the normal of the plane $y + z = a$, so pick S to be the region of the plane bounded by C
- This choice gives $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = -2$, so the integral over the whole surface is just $-2 \cdot$ (Area of the region of the plane inside C).
- It is clear that the plane $x + y = a$ cuts the sphere $x^2 + y^2 + z^2 = a^2$ in a circle (this was worked out in a previous tutorial), and the radius of this circle is $a/\sqrt{2}$ (draw the picture)
- Now, by Stokes' theorem:

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS = -2(\pi a^2/2) = -\pi a^2$$

- Caveat: The final answer is different from the solution Folland has given in the answer key, so there may be a mistake somewhere.

PROBLEM 2. *(Folland 5.8.1(ef)) Determine whether the following vector fields is the gradient of a function f, and if so, find f.*

(1) $\mathbf{F}_1(x, y, z) = (y - z)\mathbf{i} + (x - z)\mathbf{j} + (x - y)\mathbf{k}$
(2) $\mathbf{F}_2(x, y, z) = 2xy\mathbf{i} + (x^2 + \log z)\mathbf{j} + ((y + 2)/z)\mathbf{k}$

Recall that on an open convex set, there exists a C^1 function $f(x, y, z)$ such that $\mathbf{F}(x, y, z) = \nabla f$ if and only if $\nabla \times \mathbf{F} = 0$. Since $\mathbb{R}^3$ is a convex set, we just apply the theorem. For the first vector field,

$$\nabla \times \mathbf{F}_1(x, y, z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y - z & x - z & x - y \end{vmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \neq 0$$

So the first vector field is not the gradient of a function. For the second vector field,

$$\nabla \times \mathbf{F}_2(x, y, z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 2xy & x^2 + \log z & \frac{y+2}{z} \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

So there exists $f(x, y, z)$ such that $\nabla f = \mathbf{F}_2$. Let's find f. We have to solve the three equations:

$$\partial_x f = 2xy$$

$$\partial_y f = x^2 + \log z$$
$$\partial_z f = \frac{y+2}{z}$$

Integrating the first equation with respect to x gives $f(x,y,z) = x^2y + g(y,z)$. Differentiating with respect to y gives:

$$x^2 + \log z = \partial_y f = x^2 + \partial_y g \Rightarrow \partial_y g = \log z \Rightarrow g(y,z) = y\log z + h(z)$$

So $f(x,y,z) = x^2y + y\log z + h(z)$. Differentiating with respect to z gives:

$$\frac{y+2}{z} = \frac{y}{z} + h'(z) \Rightarrow h'(z) = \frac{2}{z} \Rightarrow h(z) = 2\log z + C$$

This gives a final solution:

$$f(x,y,z) = x^2y + (y+2)\log z + C$$

PROBLEM 3. *Determine whether the following vector fields are the curl of a vector field* $\mathbf{F}$, *and if so, find* $\mathbf{F}$.

- $\mathbf{G}_1(x,y,z) = (x^3 + yz)\mathbf{i} + (y - 3x^2y)\mathbf{j} + 4y^2\mathbf{k}$
- $\mathbf{G}_2(x,y,z) = (xy + z)\mathbf{i} + xz\mathbf{j} - (yz + x)\mathbf{k}$

(Thm. 5.63) Recall that on an open convex set, $\mathbf{G}(x,y,z) = \nabla \times \mathbf{F}$ for some vector field $\mathbf{F}$ if and only if $\nabla \cdot \mathbf{G} = 0$. So, by the theorem, all we have to do is compute the divergence of the vector fields.

$$\nabla \cdot \mathbf{G}_1 = 3x^2 + 1 - 3x^2 + 0 = 1 \neq 0$$

Therefore $\mathbf{G}_1$ is not the curl of a vector field. On the other hand,

$$\nabla \cdot \mathbf{G}_2 = y + 0 - y = 0$$

So $\mathbf{G}_2$ is the curl of some vector field, $\mathbf{F}$. Let's find $\mathbf{F}$. We may assume that $F_3 = 0$, then:

$$\nabla \times \mathbf{F} = \begin{pmatrix} -\partial_z F_2 \\ \partial_z F_1 \\ \partial_x F_2 - \partial_y F_1 \end{pmatrix} = \mathbf{G}_2$$

Now we can solve for some of the components:

$$F_2(x,y,z) = -xyz - z^2/2 + \varphi(x,y)$$
$$F_1(x,y,z) = xz^2/2 - \psi(x,y)$$

So combining these two results we get:

$$-yz - x = \partial_x F_2 - \partial_y F_1 = -yz + \partial_x\varphi + \partial_y\psi$$

We have the freedom now to choose $\psi(x,y) = 0$, then:

$$\partial_x\varphi = -x \Rightarrow \varphi(x,y) = -x^2/2$$

Then all together, one suitable vector field $\mathbf{F}$ is given by:

$$\mathbf{F}(x,y,z) = (xz^2/2)\mathbf{i} - (xyz + z^2/2 + x^2/2)\mathbf{j}$$